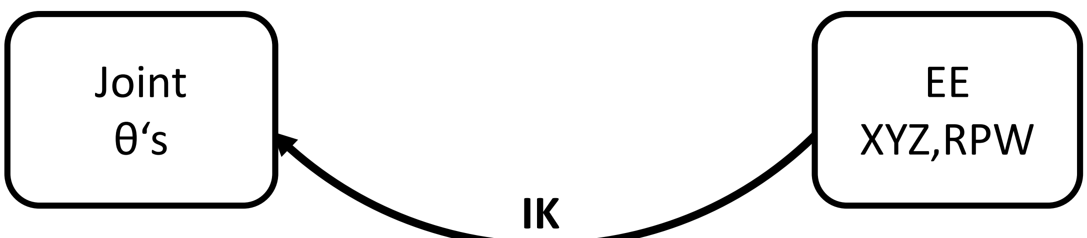
Inverse Kinematics

Chapter 3

Inverse Kinematics

- Inverse kinematics: given end effector pose, find joint positions
  - Assume each joint in the chain has 1 DoF (or is two joints, etc.)
  - More difficult than FK for serial robots
  - Easier than FK for parallel robots



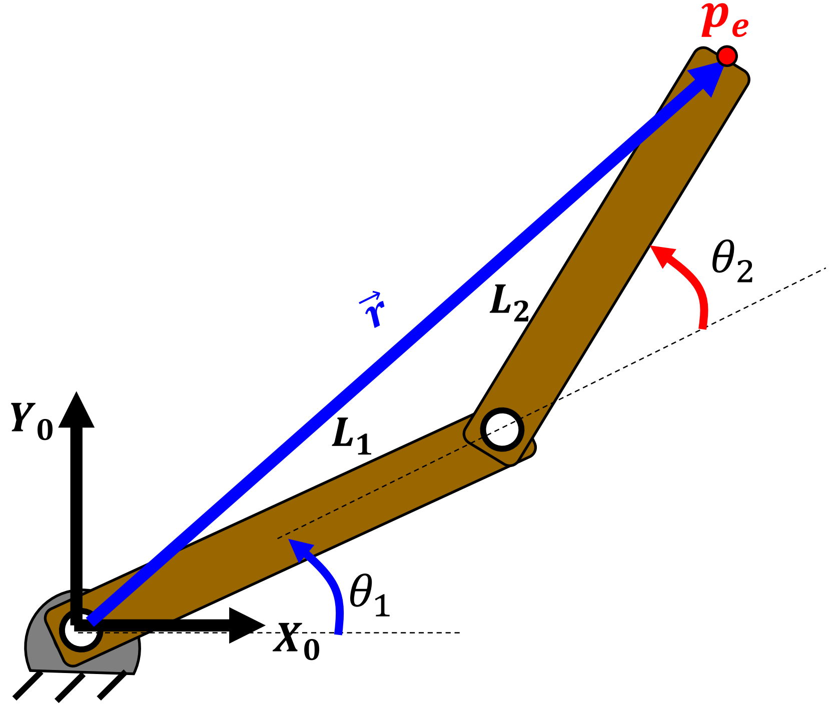
Inverse Kinematics

$$T_e^0 = \begin{bmatrix} R_e^0 & o_e^0 \\ 0 & 1 \end{bmatrix} = H$$

- Goal: Find all solutions to $T_n^0(\theta_1, \theta_2, \ldots \theta_n) = H$
- Method: take 1st 3 rows of *H* entries and solve for all d, θ
- Results: 12 equations (3x4), *n* unknowns
- Closed-form analytical solutions are ideal. Cases:
  - No solution (outside workspace)
  - One solution (edge of workspace)
  - Multiple solutions (inside workspace)
  - Impossible solutions (inside workspace)

Example – 2-Link Planar Arm

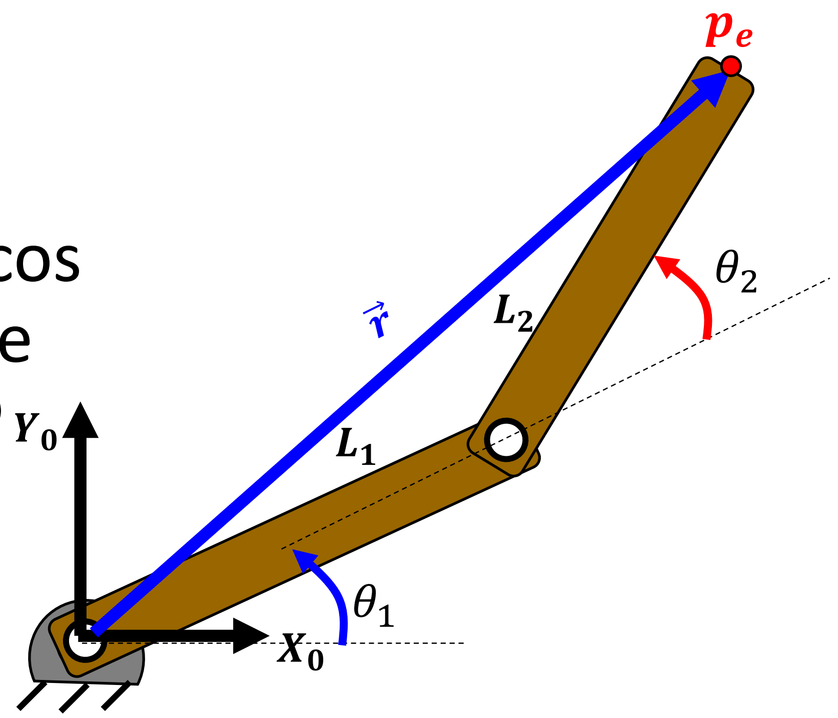
- Given $p_e = (x_e, y_e)$, find θ_1, θ_2
- Analytical solution: 2 equations, 2 unknowns



Example – 2-Link Planar Arm

$$T_e^0 = \begin{bmatrix} R_e^0 & o_e^0 \\ 0 & 1 \end{bmatrix}$$

- For θ_2:
  - Pythagorean theorem
  - Law of cosines
- For θ_1:
  - Position tangent
- Use tan instead of sin or cos for accuracy when possible
  `(In Matlab: θ=atan2(y,x))`

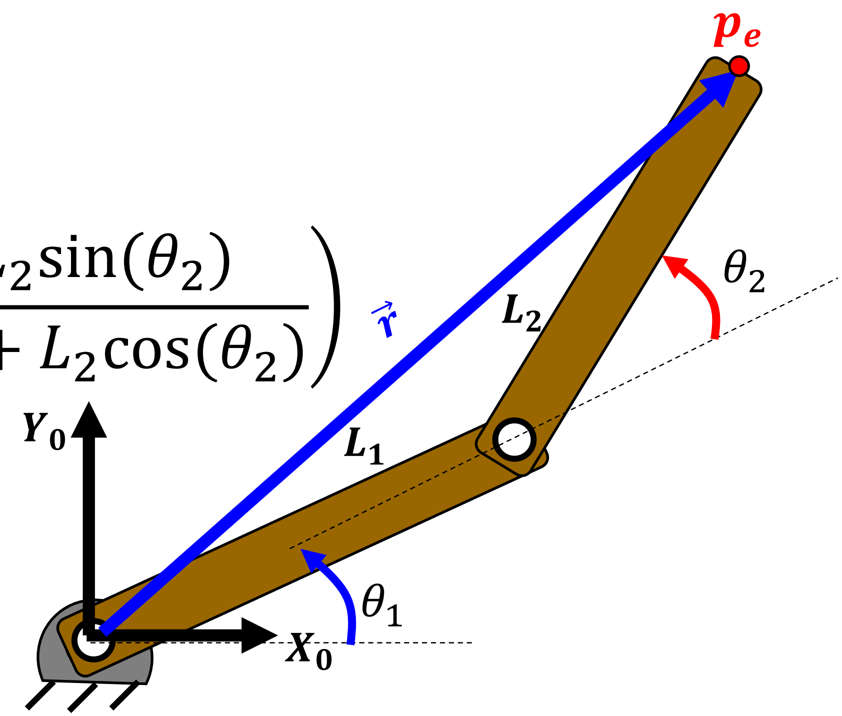


Example – 2-Link Planar Arm

$$\cos(\theta_2) = \frac{x_e^2 + y_e^2 - L_1^2 - L_2^2}{2L_1L_2} = ✌$$

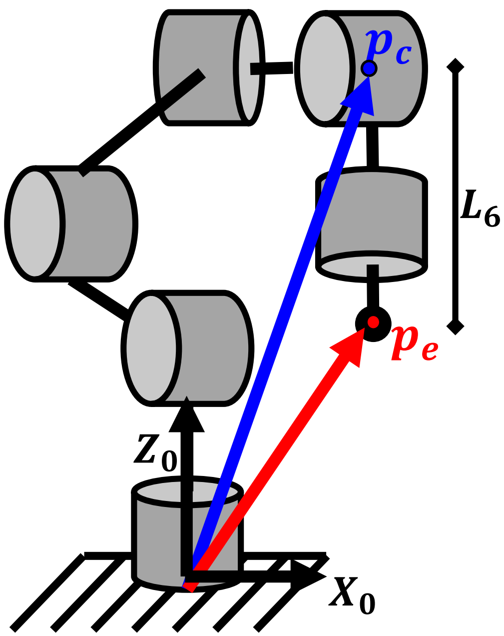
$$\theta_2 = \tan^{-1}\left(\frac{\sqrt{(1 - ✌^2)}}{✌}\right)$$

$$\theta_1 = \tan^{-1}\left(\frac{y_e}{x_e}\right) - \tan^{-1}\left(\frac{L_2\sin(\theta_2)}{L_1 + L_2\cos(\theta_2)}\right)$$



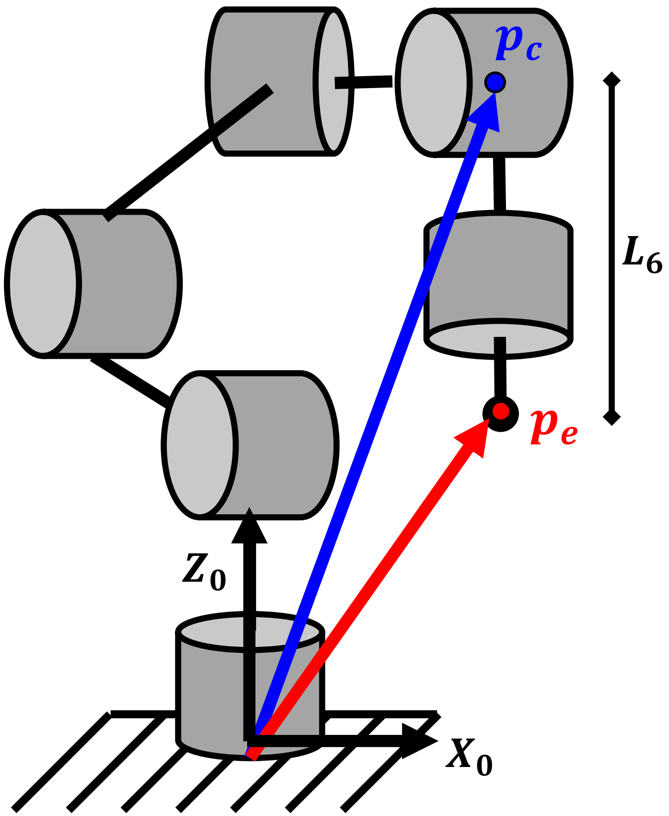
Example: 6R Serial Arm w/ 3-axis Wrist

- Most industrial robots have 6 DoF but a 3-axis intersecting wrist, so decouple equations
  - Position: $p_c \rightarrow \theta_1, \theta_2, \theta_3$
  - Orientation: $R_c^0 \rightarrow \theta_4, \theta_5, \theta_6$



Example: 6R Serial Arm w/ 3-axis Wrist

- Given: $T_e^0 = \begin{bmatrix} R_e^0 & p_e^0 \\ 0 & 1 \end{bmatrix}$ $and\ L_6$ Find: $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$

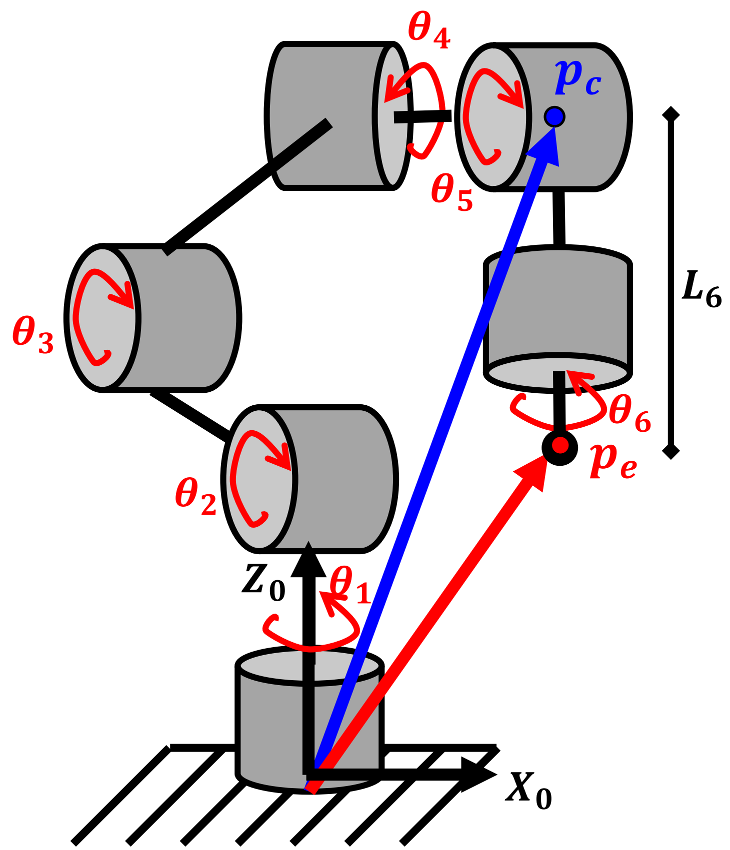


p_c^0: wrist center

p_e^0: end effector tip

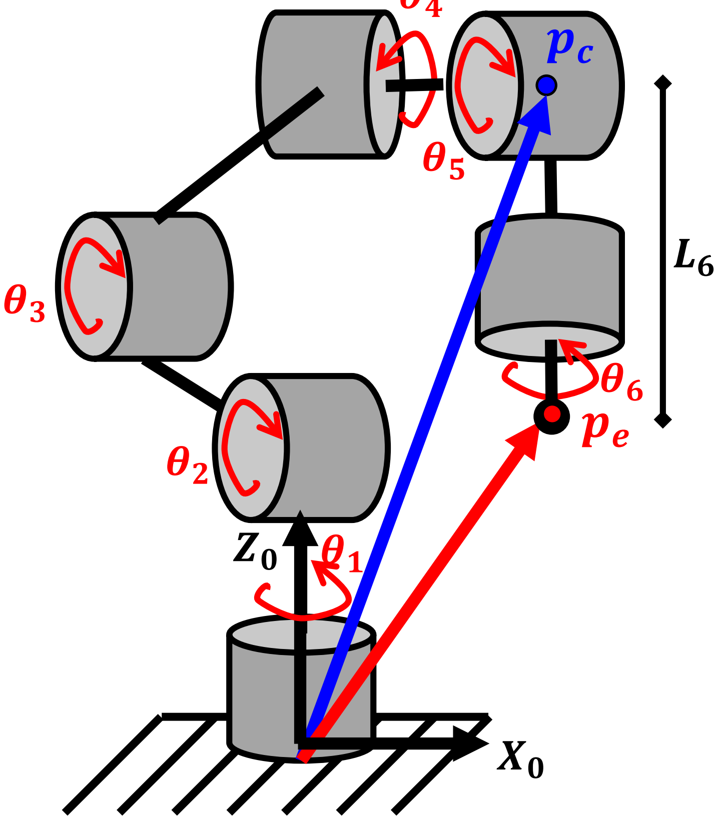
Example: 6R Serial Arm w/ 3-axis Wrist

1) Calculate $p_c^0 = p_e^0 - R_6^0 \begin{bmatrix} 0 \\ 0 \\ L_6 \end{bmatrix}$
2) Use p_c to find $\theta_1, \theta_2, \theta_3$
3) Calculate R_3^0 from $\theta_1, \theta_2, \theta_3$
4) Calculate R_6^3 from R_3^0, R_6^0
5) Use R_6^3 , ZYZ Euler angles, DH to find $\theta_4, \theta_5, \theta_6$



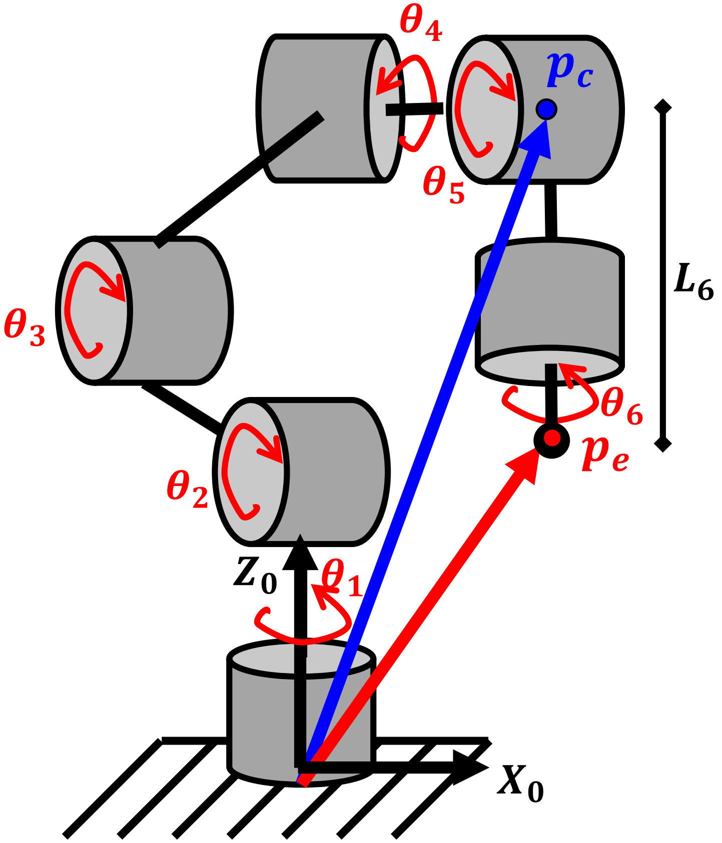
Example: 6R Serial Arm w/ 3-axis Wrist

1) Calculate $p_c^0 = p_e^0 - R_6^0 \begin{bmatrix} 0 \\ 0 \\ L_6 \end{bmatrix}$



Example: 6R Serial Arm w/ 3-axis Wrist

2) Use p_c to find $\theta_1, \theta_2, \theta_3$
   - Project onto $x_0 y_0$ plane
   - Solve using trig: $\theta_1 = \tan^{-1}\left(\frac{y_e}{x_e}\right)$
   - Solve for θ_2, θ_3 using more trig



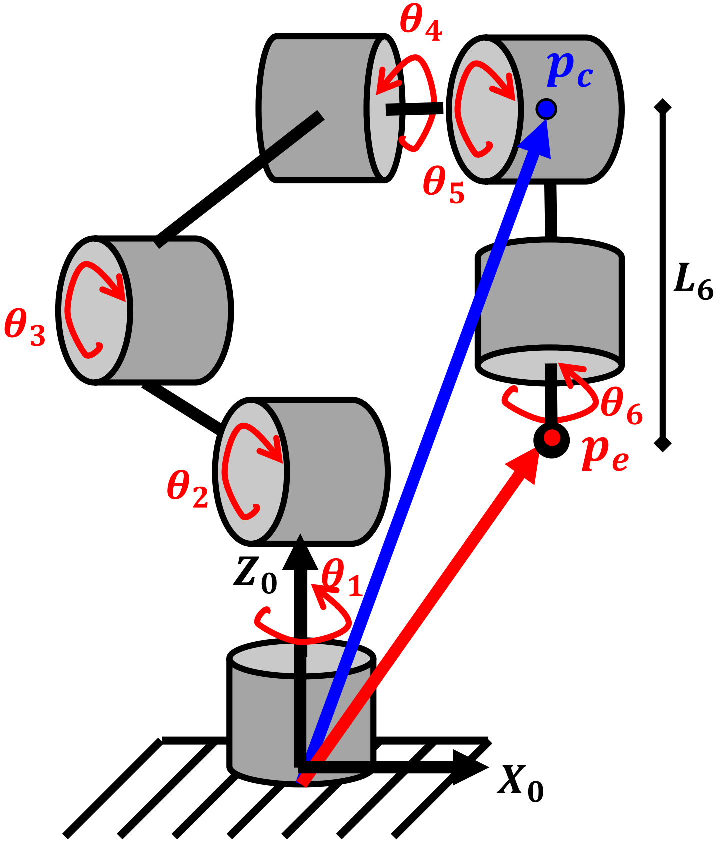
Example: 6R Serial Arm w/ 3-axis Wrist

3) Calculate R_3^0 $\theta_1, \theta_2, \theta_3$

- Use $\theta_1, \theta_2, \theta_3$
- $R_3^0 = R_1^0 R_2^1 R_3^2$

$$= R_{z\theta_1}(R_{x\pi/2} R_{z\theta_2}) R_{z\theta_3}$$

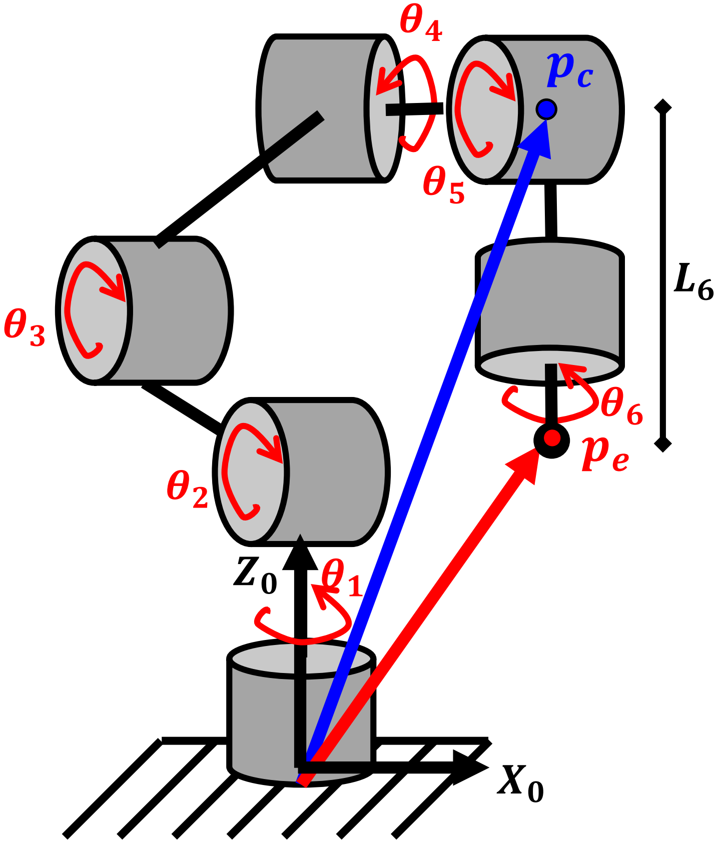
$$R_3^0 = \begin{bmatrix} c_1 c_{23} & c_1 s_{23} & s_1 \\ s_1 c_{23} & s_1 s_{23} & -c_1 \\ s_{23} & c_{23} & 0 \end{bmatrix}$$



Example: 6R Serial Arm w/ 3-axis Wrist

4) Calculate R_6^3

$$R_6^3 = (R_3^0)^T R_6^0$$



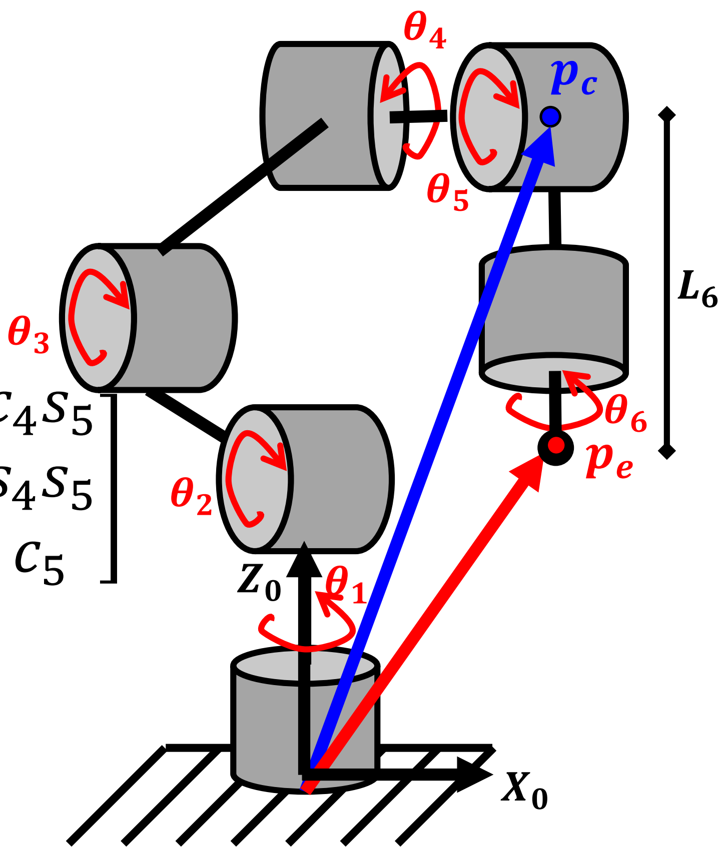
Example: 6R Serial Arm w/ 3-axis Wrist

5) Find $\theta_4, \theta_5, \theta_6$

- Use R_6^3 and ZYZ Euler angles
  - $R_6^3 = R_{z\theta_4} R_{y\theta_5} R_{z\theta_6}$
  - Eqns 2.28-2.33
- Use R_6^3 and DH parameters

$$R_6^3 = \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - c_4 s_6 & c_4 s_5 \\ s_4 c_5 c_6 + c_4 s_6 & -s_4 c_5 s_6 + c_4 c_6 & s_4 s_5 \\ -s_5 c_6 & s_5 s_6 & c_5 \end{bmatrix}$$

- Set equations equal and solve



Example: 6R Serial Arm w/ 3-axis Wrist

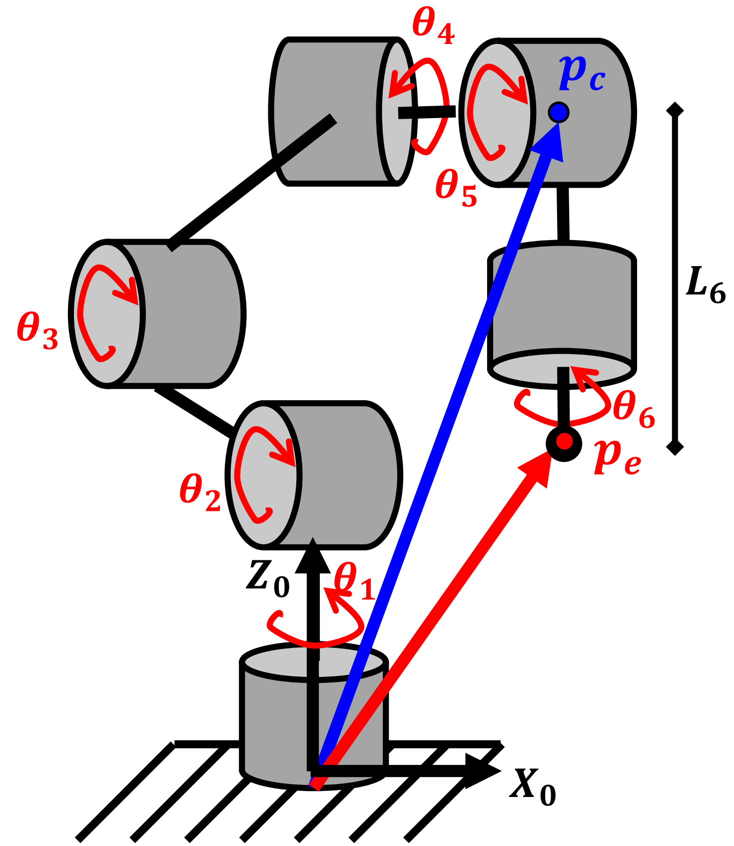
1) Calculate $p_c^0 = p_e^0 - R_6^0 \begin{bmatrix} 0 \\ 0 \\ L_6 \end{bmatrix}$

2) Use p_c to find $\theta_1, \theta_2, \theta_3$

3) Calculate R_3^0 from $\theta_1, \theta_2, \theta_3$

4) Calculate R_6^3 from R_3^0, R_6^0

5) Use R_6^3, ZYZ Euler angles, DH to find $\theta_4, \theta_5, \theta_6$

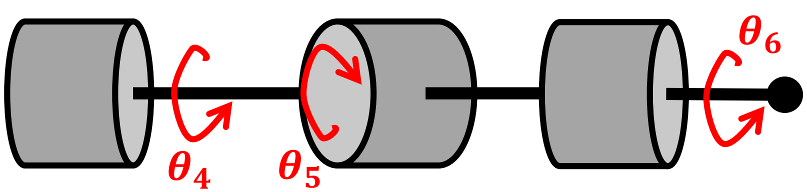


Results: eq'ns 3.64-3.69 in book

Yields 16 solutions

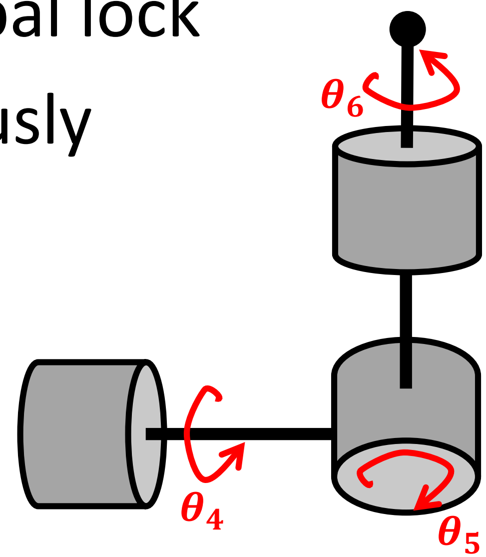
Singularities

- Beware singularities!
- Joints 4 and 6 can align in gimbal lock
- Wrist can't rotate instantaneously
- DoF lost



SINGULAR

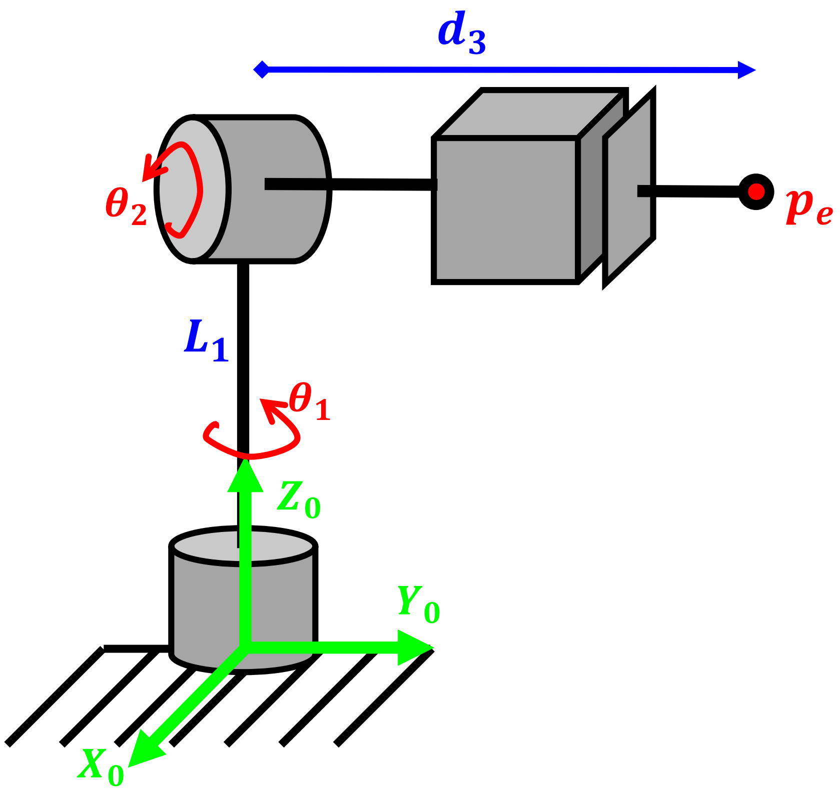
∞ solutions



NONSINGULAR

2 solutions

Example – RRP Stanford Manipulator



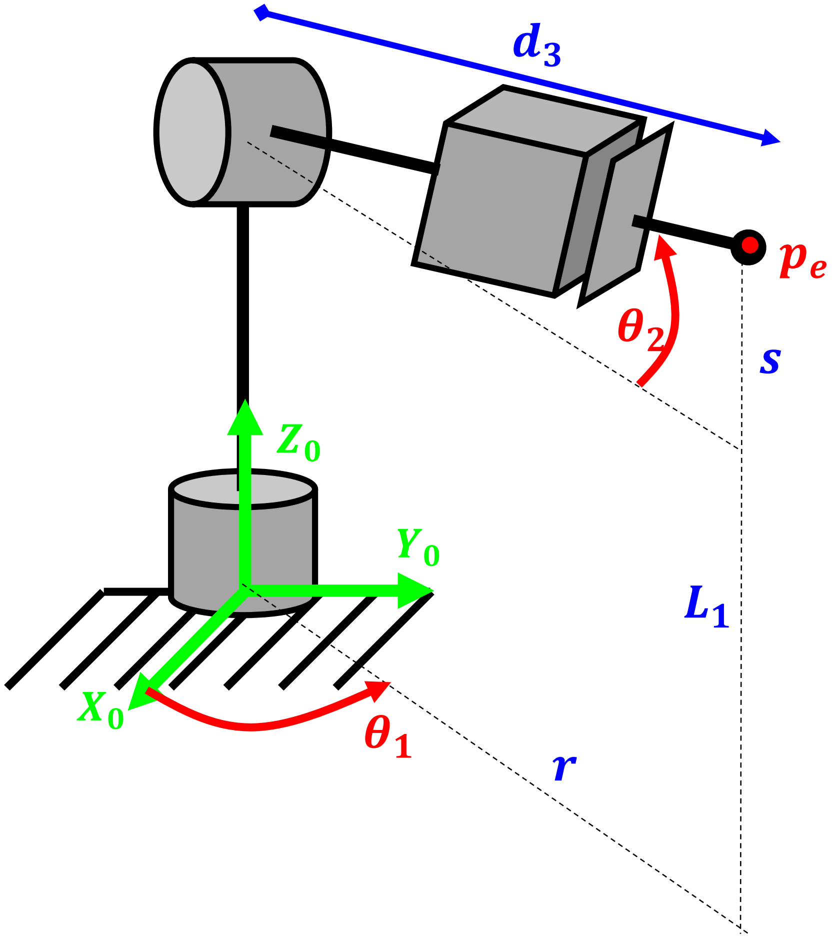
Example – RRP Stanford Manipulator

- Treat similarly to RRR

1) Find θ_1 from $x_0 y_0$ plane

2) Find θ_2 using trig

3) Find d_3 using Pythagorean Theorem

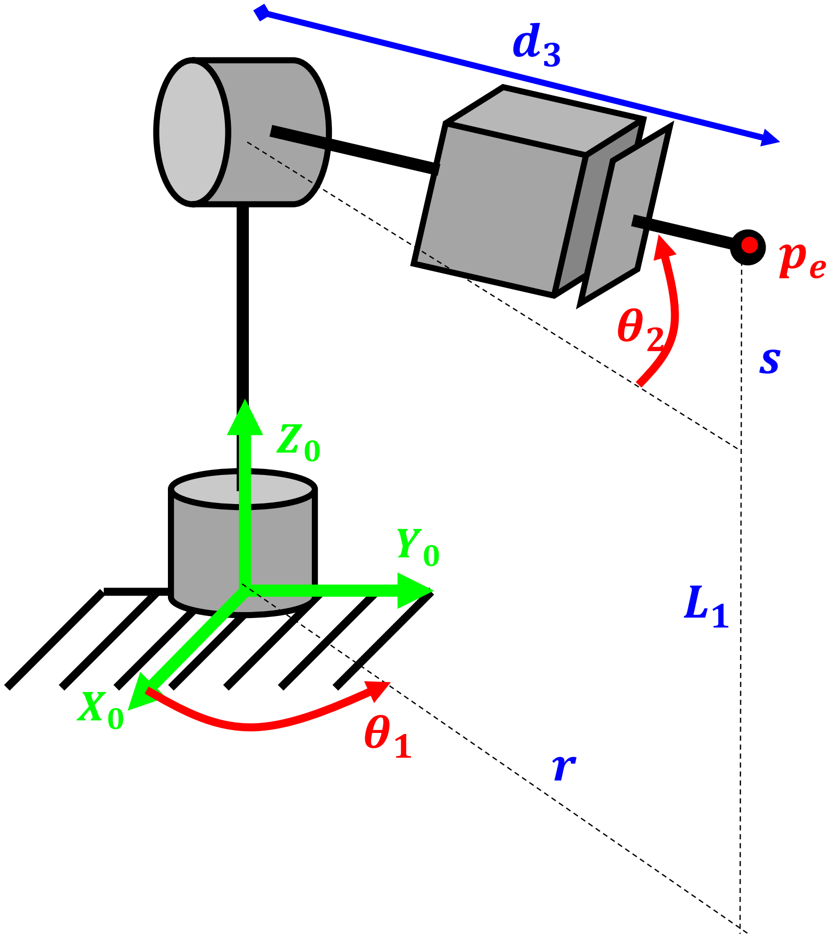


Example – RRP Stanford Manipulator

- Treat similarly to RRR

1) Find θ_1 from $x_0 y_0$ plane

$$\theta_1 = \tan^{-1}\left(\frac{-x_e}{y_e}\right)$$



Example – RRP Stanford Manipulator

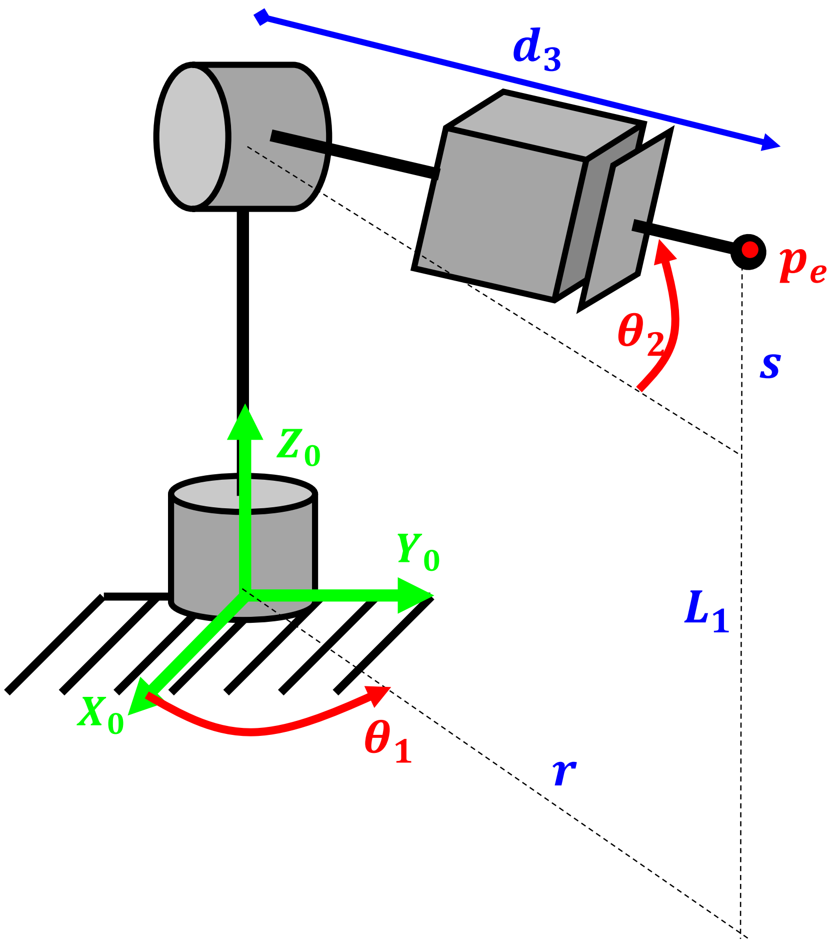
- Treat similarly to RRR

1) Find θ_1 from $x_0 y_0$ plane

$$\theta_1 = \tan^{-1}\left(\frac{-x_e}{y_e}\right)$$

2) Find θ_2 using trig

$$\theta_2 = \tan^{-1}\left(\frac{s}{r}\right) = \tan^{-1}\left(\frac{z_e - L_1}{\sqrt{x^2 + y^2}}\right)$$



Example – RRP Stanford Manipulator

- Treat similarly to RRR

1) Find θ_1 from $x_0 y_0$ plane

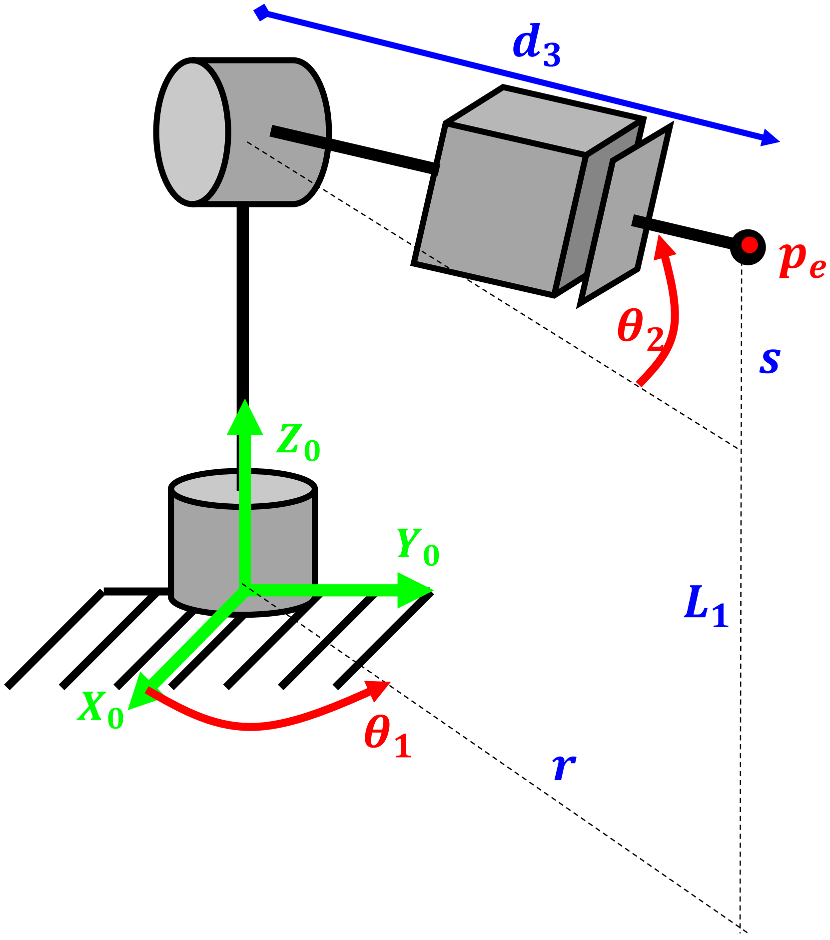
$$\theta_1 = \tan^{-1}\left(\frac{-x_e}{y_e}\right)$$

2) Find θ_2 using trig

$$\theta_2 = \tan^{-1}\left(\frac{s}{r}\right) = \tan^{-1}\left(\frac{z_e - L_1}{\sqrt{x^2 + y^2}}\right)$$

3) Find d_3 using Pythagorean Theorem

$$d_3 = \sqrt{r^2 + s^2}$$



General Notes

- A manipulator is solvable if ALL sets of joint variables for any pose can be found
- All single-chain 6-DOF robots (RP combos)
are solvable at least numerically (Matlab!)
Use Jacobian and resolved rates to min joint motion
- Analytical solutions for the 6R robot are only possible with a 3-axis intersecting wrist

6R, 5RP → 16 solutions

4R2P → 8 solutions

3R3P → 2 solutions

Redundant Robots

- Redundant robots have more DoF than needed, so more unknowns than equations

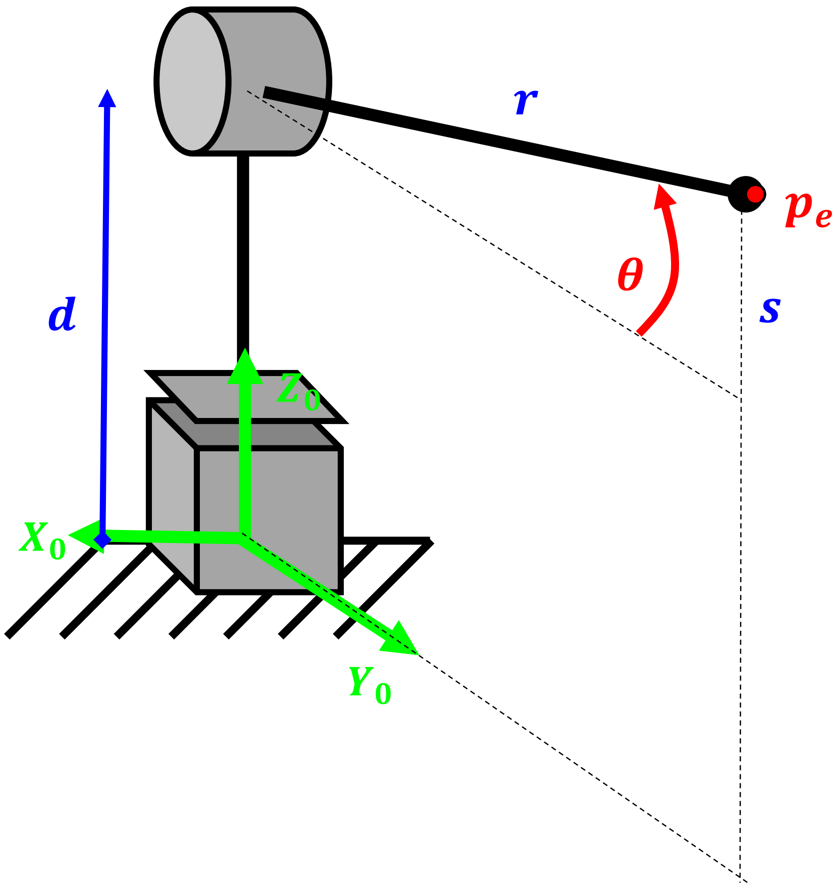
  ∞ solutions inside workspace

  1 solution   on workspace edge

  0 solutions outside workspace

Example – PR Manipulator

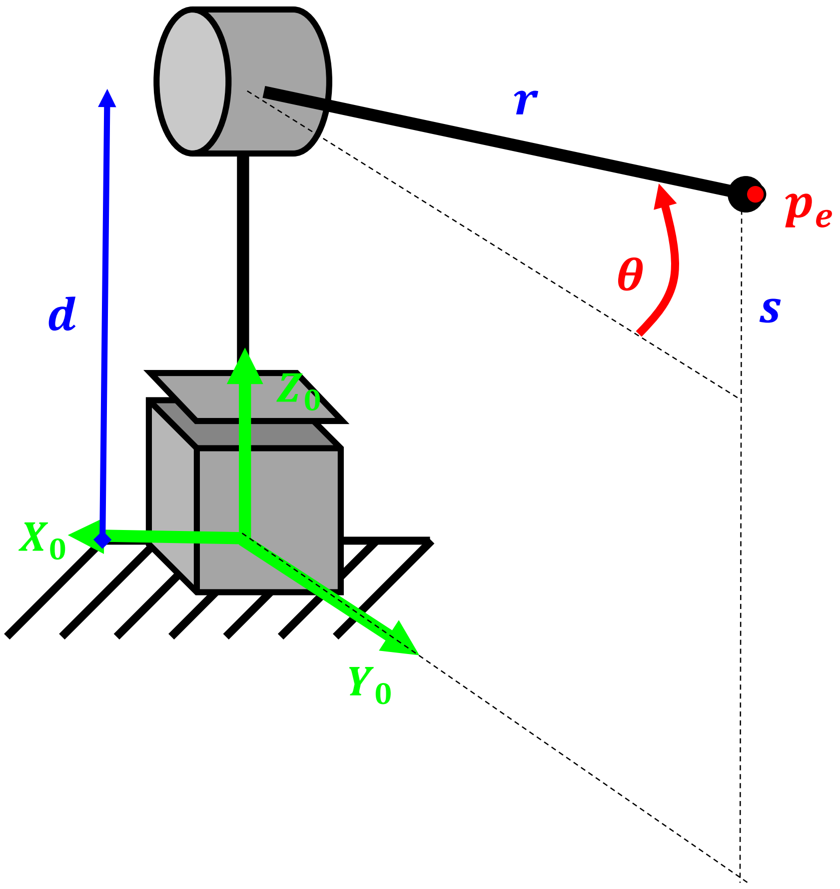
- Given: r, p_e
- Find: d, θ



Example – PR Manipulator

1) Find θ from r, y_e

2) Find d from θ, z_e



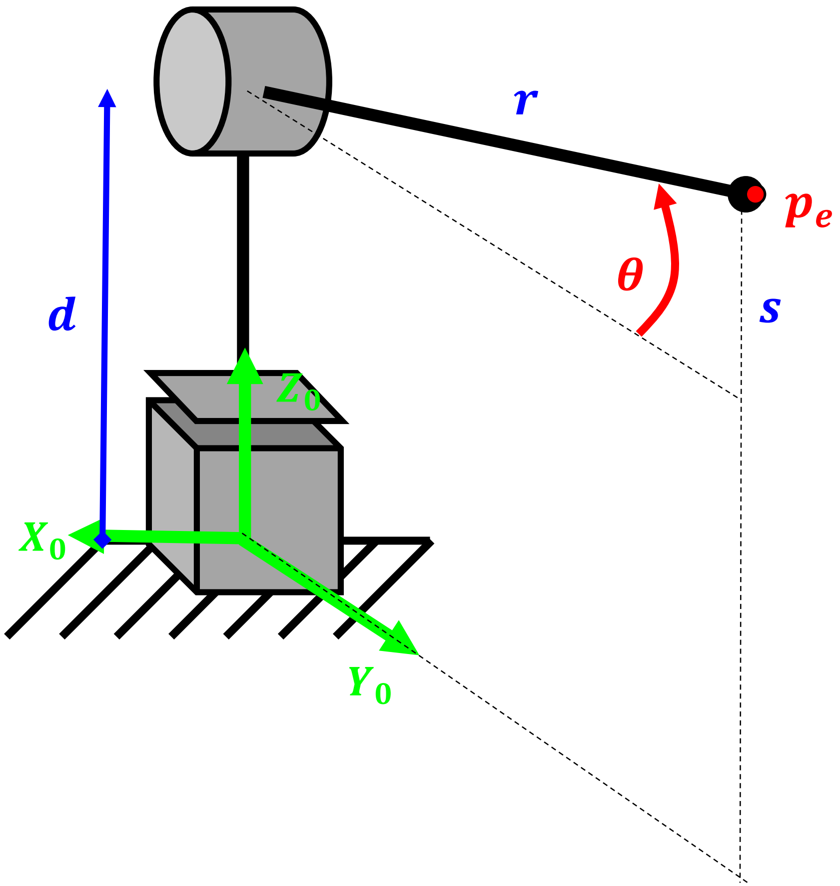
Example – PR Manipulator

1) Find θ from r, y_e

$$\theta = \tan^{-1}(\frac{\sqrt{(r^2 - y_e^2)}}{y_e})$$

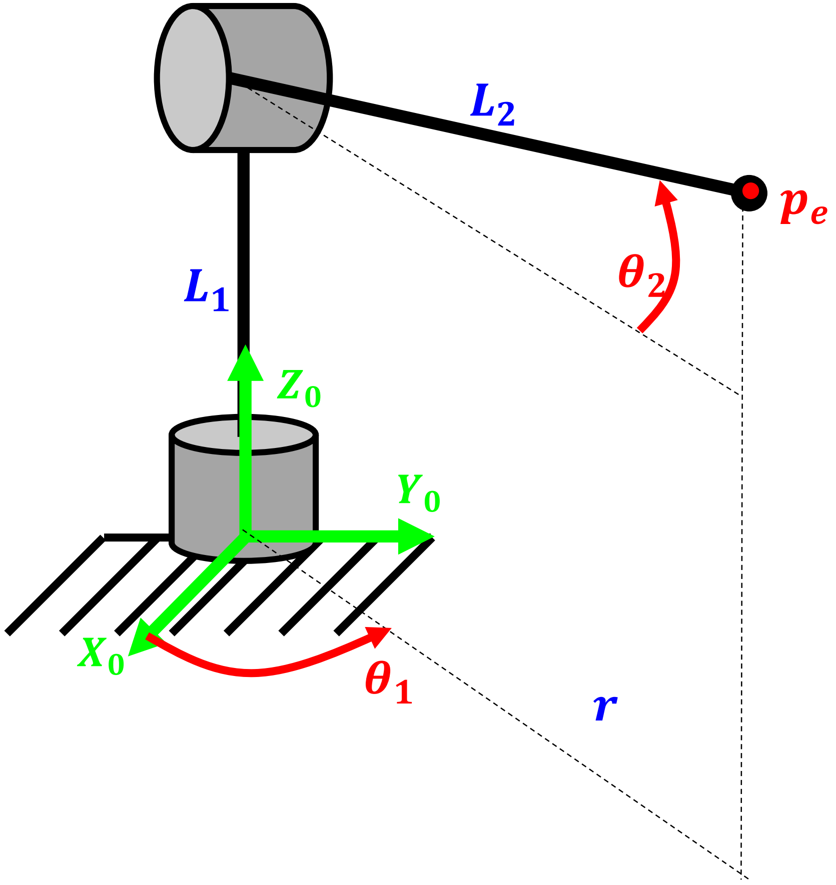
2) Find d from θ, z_e

$$d = z_e - r sin(\theta)$$



Example – RR Manipulator

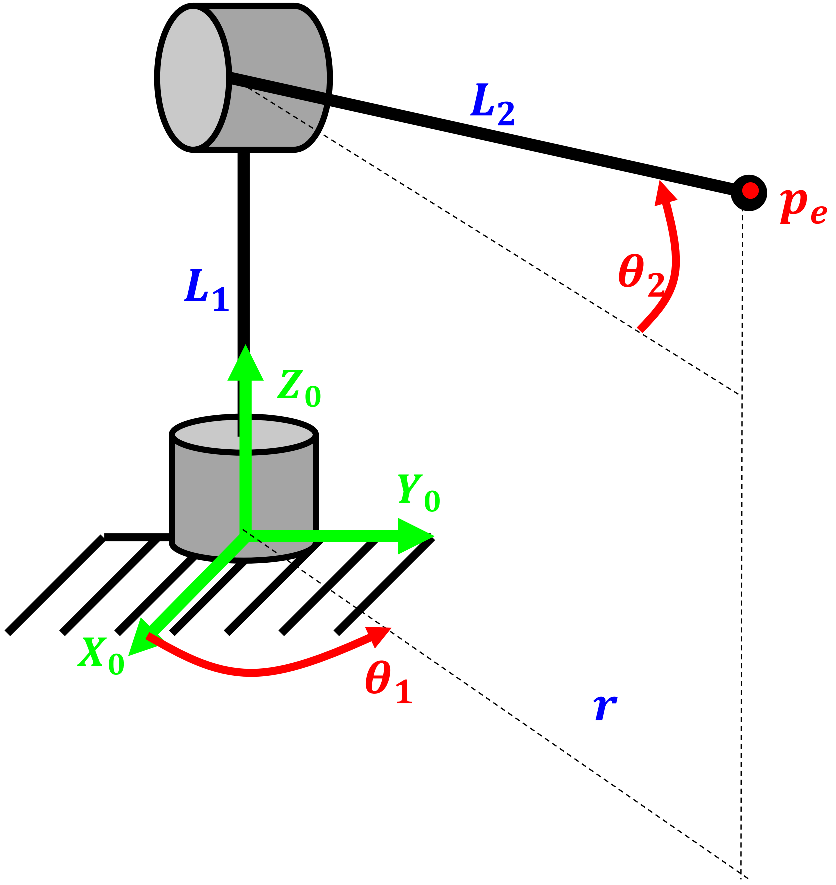
- Given: L_1, L_2, p_e
- Find: θ_1, θ_2



Example – RR Manipulator

1) Find θ_1 from x_e, y_e

2) Find θ_2 from z_e



Example – RR Manipulator

1) Find θ_1 from x_e, y_e

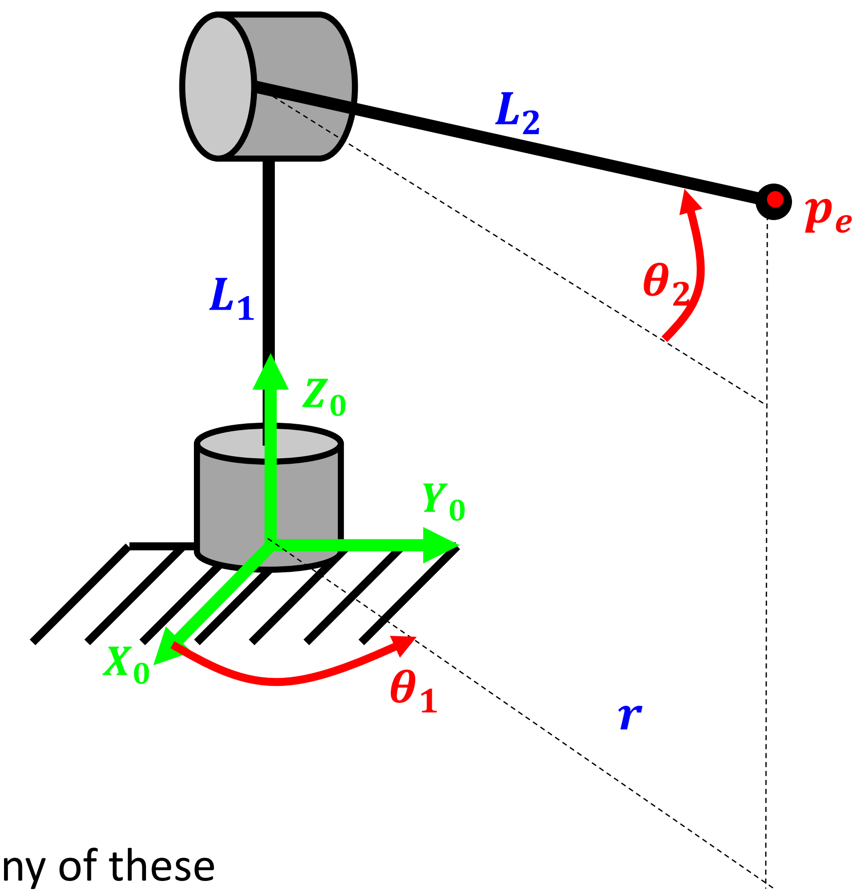
$$\theta_1 = \tan^{-1}(\frac{y_e}{x_e})$$

2) Find θ_2 from z_e

$$\theta_2 = \sin^{-1}(\frac{z_e - L_1}{L_2})$$

$$\theta_2 = \cos^{-1}(\frac{\sqrt{x^2 + y^2}}{L_2})$$

$$\theta_2 = \tan^{-1}(\frac{z_e - L_1}{\sqrt{x^2 + y^2}})$$



Any of these are acceptable on an exam